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A REVIEW OF NUMERICAL ANALYSIS OF TUNNELS IN DISCONTINUOUS ROCK MASSES

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SUMMARY

Rock masses in nature contain numerous discontinuities in the form of cracks, joints, faults, bedding planes, etc. Therefore, various continuum equivalent models of discontinuum models have been proposed and used to assess the stability of rock tunnels since the beginning of 1970. In this article, a brief but comprehensive review of numerical analysis of tunnels in discontinuous rock masses is presented. Firstly, various discrete models are briefly described and their fundamental features are summarised. Then, equivalent continuum models are described and their fundamental assumptions are presented. And then a brief outline of hybrid models proposed in literature is given. Finally, some recommendations as to how to use these models in practice are made. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: discontinuum; continuum; contact discrete model; equivalent model; hybrid model

1. INTRODUCTION

Discontinuum is distinguished from continuum by the existence of contacts or interfaces between the discrete bodies that comprise the system. Relative sliding or separational movements in such localized zones present an extremely difficult problem in mechanical modelling and numerical analysis. The formulation for representing contacts is very important when a system of interacting blocks is considered, and it has been receiving a considerable interest among researchers.

There are two fundamental theoretical models, namely Hertz's model and Mindlin's model, for modelling contacts.[1] However, these models are restricted to a very simple geometry and the elastic behaviour of adjacent materials. Since the configuration of contacts and the mechanical behaviour of adjacent materials are generally complex, the experimental technique is probably the only way to deal with contact problems. In this respect, the direct shear test technique is widely used to characterize the behaviour of contacts, interfaces and rock discontinuities.

There have been mainly three kinds of modelling to interpret and to utilise the responses measured in direct shear tests (Figure 1):

(1) *Force–displacement-type modelling*:[2,3] In this case, contacts are assumed to have a zero thickness without an explicit definition of contact area A_c. The responses measured in direct shear tests are directly used in numerical representations.

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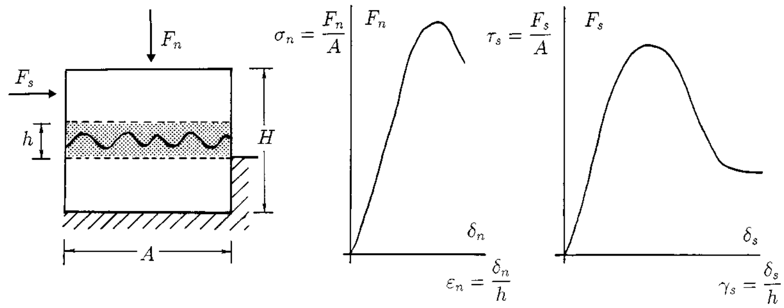

Figure 1. Mechanical interpretation of normal and shear responses of rock discontinuities

(2) *Stress–displacement-type modelling*:[4] This type of modelling is probably the most widely used approach in numerical analyses. The contacts are again assumed to have a zero-thickness and the responses measured in direct shear tests are directly used in numerical representations.
(3) *Stress–strain type-modelling* (*band type modelling*):[5–8] Contacts, rock discontinuities and interfaces are considered as bands with a finite thickness. The thickness of the bands is related to the thickness of shear-bands observed in tests or in nature, or the height of asperities along the plane.

On the basis of these mechanical models, numerical models on tunnels in discontinuous rock masses can be categorized into three groups:

(1) discrete models,
(2) equivalents models, and
(3) hybrid models.

In discrete modelling, discontinuities are modelled as planes having normal and shear stiffnesses[4] or thin bands.[6,8,9]

Equivalent continuum modelling starts with the first attempt by Singh.[10] His model is based upon the theory proposed by Hill[11] for composite materials. Most of the follow-up models basically follow the foot steps of Singh. Some of the later variations of this model are merely concerned with how to define additional strain due to discontinuities over the representative elementary volume (REV) and how to relate the stress field of discontinuities to that acting on the REV. The models now available in rock mechanics are:

(1) equivalent elastic compliance model;[10]
(2) crack tensor model;[12–14]
(3) damage model;[15–17]
(4) micro-structure models;[18,19]
(5) homogenization technique.[20]

Hybrid models are a combination of discrete and equivalent models within the same scheme. There are several combinations depending upon the chosen type of numerical schemes.

2. DISCRETE MODELS

By reviewing the literature, it can be found that during the last three decades the limiting equilibrium analysis[21] and some numerical analysis methods such as the Finite Element Method (FEM),[4–9] distinct element method (DEM)[22] and discontinuous deformation analysis (DDA)[23] have been developed for the analysis of problems involving discontinuities in rock mechanics. A brief review of these techniques with a particular emphasis on tunnelling is given in the followings.

2.1. Discrete Finite Element Method (DFEM)

Finite element techniques using contact, joint or interface elements, have been developed for representing discontinuities between blocks in rock masses. The most simple approach for representing joints is the contact element,[2] which was originally developed for bond problems between steel bars and concrete (Figure 2). The contact element is a two-noded element having normal and shear stiffnesses. This model is recently improved to model block systems by Aydan–Mamaghani[24,25] by assigning a finite thickness to contact element and employing an up-dated Lagrangian scheme to deal with large block movements (Figure 3). The contact element can easily deal with sliding and separational movements.

Goodman *et al.*[4] proposed a four-noded joint element for joints (Figure 4). This model is simply a four-noded version of the contact element of Ngo and Scordelis[2] and it has the following characteristics. In a two-dimensional domain, joints are assumed to be tabular with zero thickness. They have no resistance to the net tensile forces in the normal direction, but they have high resistance to compression. Joint elements may deform under normal pressure, especially if there are crushable asperities. The shear strength is presented by a bi-linear Mohr–Coulomb envelope. The joint elements are designed to be compatible with solid elements.

Ghaboussi *et al.*[5] proposed a four-noded interface element for joints (Figure 5). This model is a further improvement of the joint element by assigning a finite thickness to joints. Zienkiewicz and Pande[27] modified the formulation of rectangular elements to model joints and introduced an

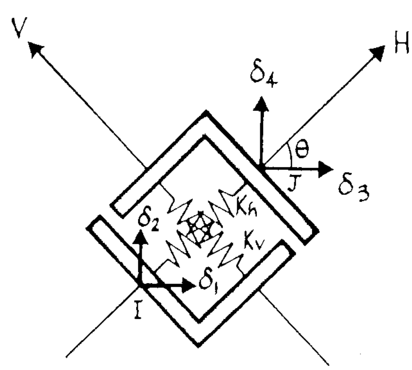

Figure 2. Contact element of Ngo and Scordelis

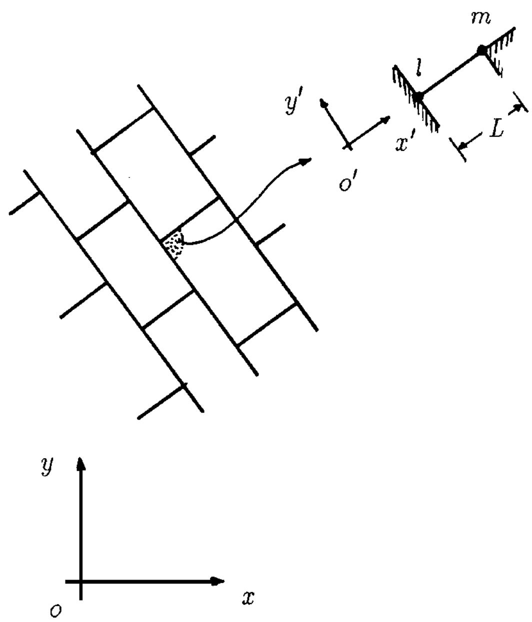

Figure 3. Contact element of Aydan–Mamaghani

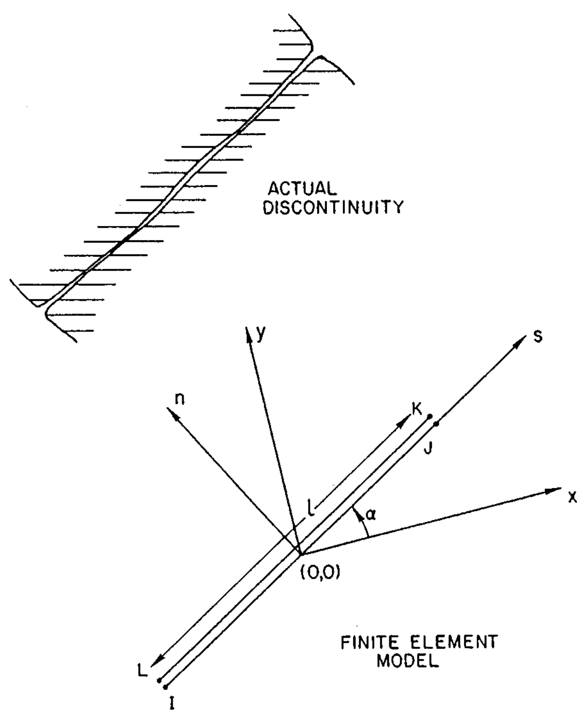

Figure 4. Joint element of Goodman

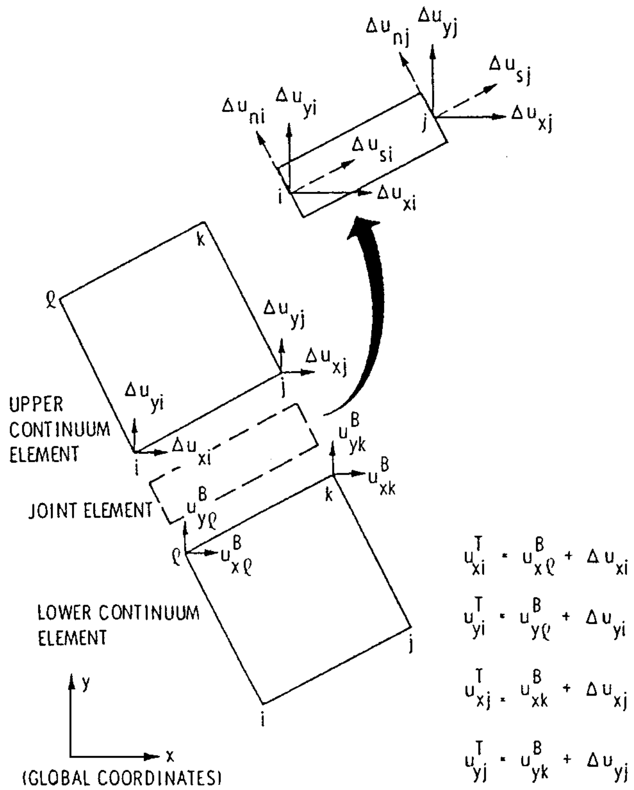

Figure 5. Joint element of Ghaboussi

elastic-visco-plastic-type constitutive law for joints. The thin layer element proposed by Desai *et al.*[6] is also similar to that of Ghaboussi.

These models are widely used to model tunnels in fractured and jointed media.[10,27–30]

2.2. Distinct Element Method (Cundall's model)

Distinct element method (rigid block models) for jointed rocks was developed by Cundall[2] in 1971. In Cundall's model problems are treated as dynamic ones. It is assumed that the contact force is produced by the action of springs which are applied whenever a corner penetrates an edge (Figure 6). Normal and shear stiffnesses were introduced between the respective forces and displacements in his original model. Furthermore, to account for slippage and separation of block contacts, he also introduced the law of plasticity. For the simplicity of calculation of contact forces due to the overlapping of the block, he assumed that the blocks do not change their original configurations. To solve the equations of the whole domain, he never assembled the equilibrium equations of blocks into a large equation system but solved them through a step-by-step procedure which he called marching scheme. His solution technique has two main merits:

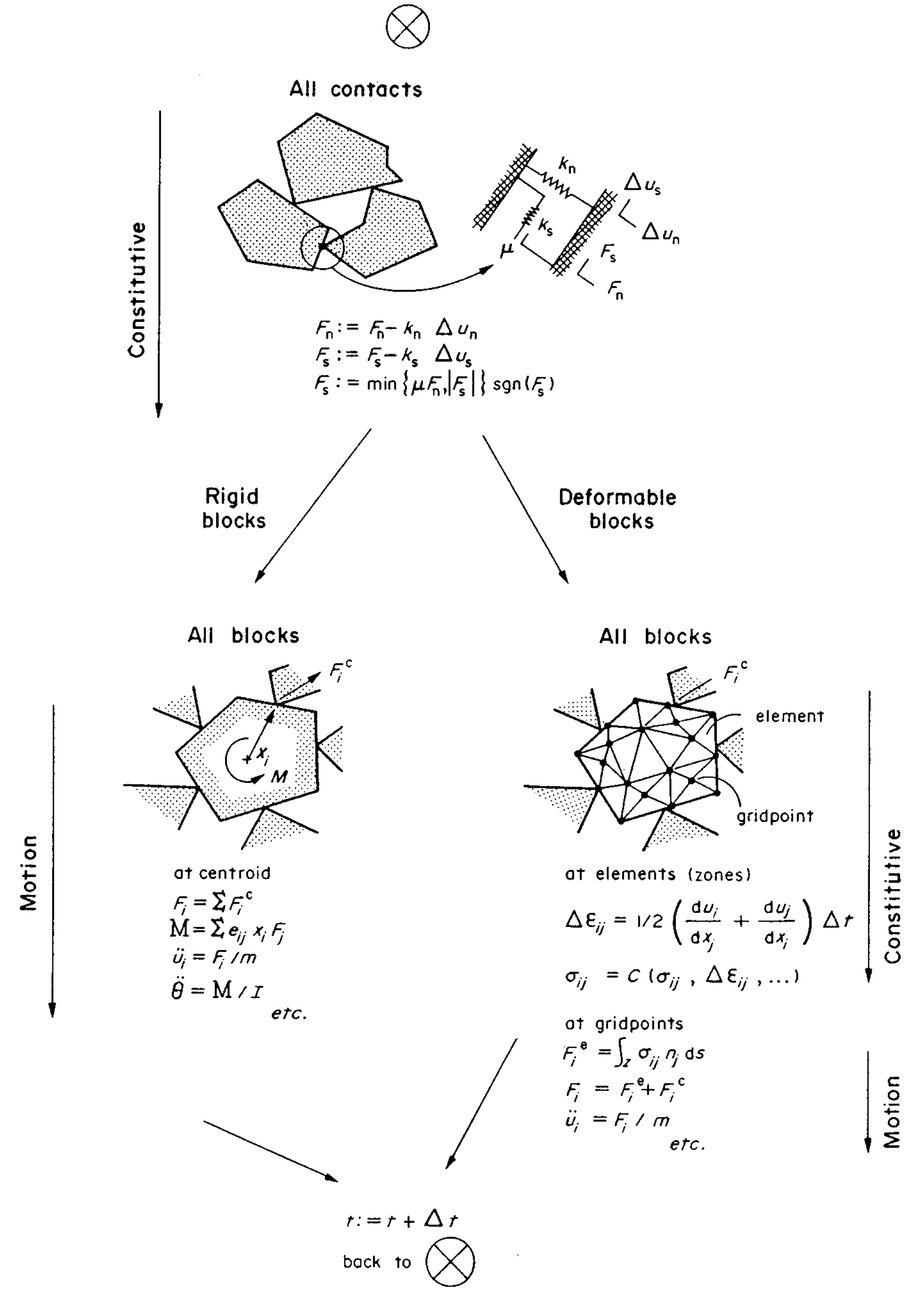

Figure 6. Calculation scheme for the Distinct Element Method (DEM)

(1) Storage memory of computers can be small (note that computer technology was not so advanced during the late 1960's), therefore, it could run on a microcomputer.
(2) The separation and slippage of contacts can be easily taken into account since the global matrix representing block connectivity is never assembled. If a large assembled matrix is used, such matrix will result in zero or very nearly zero diagonals which subsequently cause singularity or ill conditioning of the matrix.

Since the governing equation is of hyperbolic type, the system could not become stabilized even for static cases unless a damping is introduced into the equation system. In recent years he improved the original model by considering the deformability of intact blocks and their elasto-plastic behaviour.[31,32] Cundall's model has been actively used in tunnel design by the NGI group[33,34] in recent years.

2.3. Displacement Discontinuity Method (Crouch and Starfield)

This technique is generally used together with the Boundary Element Method (BEM). The discontinuities are modelled as a finite length segment in an elastic medium with a relative displacement.[35] In other words, the discontinuities are treated as internal boundaries with prescribed displacements. As an alternative approach to the technique of Crouch and Starfield,[35] Crotty and Wardle[36] use interface elements to model discontinuities and the domain is discretized into several sub-domains.

2.4. Discontinuous Deformation Analysis Method (Shi's method)

Shi[23] proposed a method called Discontinuous Deformation Analysis (DDA). Intact blocks were assumed to be deformable and are subjected to constant strain and stress due to the order of the interpolation functions used for the displacement field of the blocks. In the original model, the inertia term was neglected so that the damping becomes unnecessary. For dynamic problems, although damping is not introduced into the system, the large time steps used in the numerical integration in time-domain results in artificial damping. It should be noted that this type of damping is due to the integration technique for time domain and has nothing to do with the mechanical characteristics of rock masses (i.e. frictional properties). Although the fundamental concept is not very different from Cundall's model, the main difference results from the solution procedure adopted in both methods. In other words, the equation system of blocks and its contacts are assembled into a global equation system in Shi's approach. Recently Ohnishi *et al.*[37] introduced an elasto-plastic constitutive law for intact blocks and gave an application of this method to tunnelling.

3. EQUIVALENT METHODS

The main characteristics of the models are described in this section.

3.1. Equivalent Elastic Compliance Model (EECM): (Singh's model)

Singh's model is based on the theory proposed by Hill[11] for composite materials, and the elastic constitutive law of the rock mass is obtained by making the following assumptions (Figure 7):

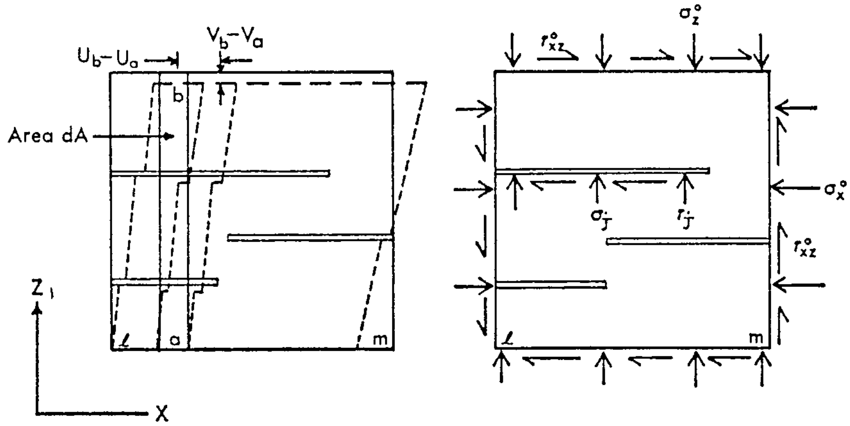

Figure 7. Mechanical model of the Representative Elementary Volume (REV) for Singh's elastic compliance model

(1) Discontinuities are distributed in sets in the rock mass.
(2) The geometry of discontinuities (area and orientation) are known.
(3) The constitutive law of discontinuities are expressed in terms of relative normal and shear displacements and applied shear and normal stresses, and the shear and normal stiffnesses are used to express the behaviour of discontinuities.
(4) The stress tensor acting on discontinuities is related to that acting on the representative volume through a tensor called stress-concentration tensor.
(5) The strain tensor of the representative element is a linear sum of the strain tensor of the intact rock and the additional strain tensor due to discontinuities.
(6) The volume of discontinuities is assumed to be negligible as compared with that of the rock so that the stress tensor acting on the representative volume of rock mass is the same as that on the intact rock.

The constitutive law derived in a local co-ordinate system is then transformed to that in the global co-ordinate system. The formulations given by Goodman[38] and Amadei and Goodman[39] are the simplified form of Singh's model.

Application of these models to tunnels in jointed media with a cross-continuous pattern and intermittent pattern are given and compared with a discrete model. This model is the first equivalent model to be applied to tunnelling.

3.2. Crack Tensor Model (CTM)

This model proposed by Oda[12,13] for rock masses follows basically the same steps of Singh's model in order to obtain the elastic constitutive law of the rock mass. The main differences are as follows:

(1) Constitutive law is directly derived in a global co-ordinate system.
(2) The geometry of discontinuities (area and orientation) are represented by a series of even order tensors (up to fourth-order tensors).

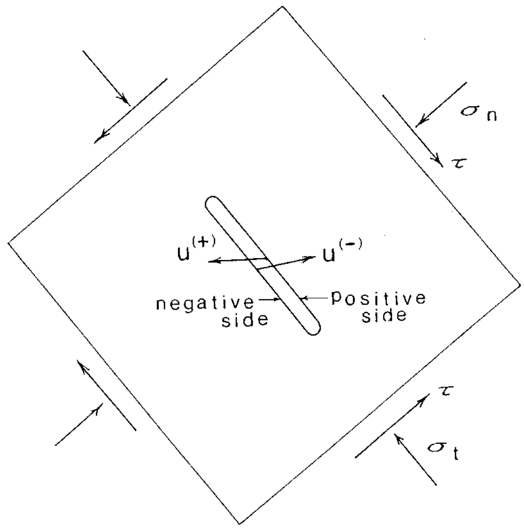

Figure 8. Applied stress and displacements at crack surfaces for Oda's Crack Tensor Model (CTM)

(3) The additional strain tensor due to discontinuities is determined from a procedure utilising an analogy to the theoretical solutions for penny shape or eliptical inclusions embedded in elastic medium used in the linear elastic fracture mechanics[40] (Figure 8).

The application of this model to tunnels, particularly, branching tunnels are described in a recent paper by Oda *et al.*[14]

3.3. Damage Model (DM)

Damage model is based upon the theory proposed by Kachanov[41] for creeping metals. It is elaborated by Murakami[15] by introducing a second-order tensor called the damage tensor and it is applied to a rock mass by Kyoya[17] and Kawamoto *et al.*[16] Assumptions 1 and 2 of Singh are also the same in this model. However, this model differs from other models and it is based on the following additional assumptions:

(1) The discontinuities are assumed to be not transmitting any stress across which implies the discontinuities has no stiffness at all.
(2) The stresses assumed to be acting only on the intact parts which implies the parallel connection principle for stress field. The average stress (Cauchy stress) is related to the stress (net stress or intensified stress) on the intact part through the second order damage tensor which represents a tensorial area reduction in the mass (Figure 9).
(3) The strain tensor of the representative elementary volume is the same as that of the intact rock.
(4) The constitutive law is introduced between the net stress tensor and the strain tensor.

It should be noted that this model could not directly used for throughgoing discontinuity sets because of Assumption 1. Nevertheless, Kyoya[17] introduces some coefficients for normal and

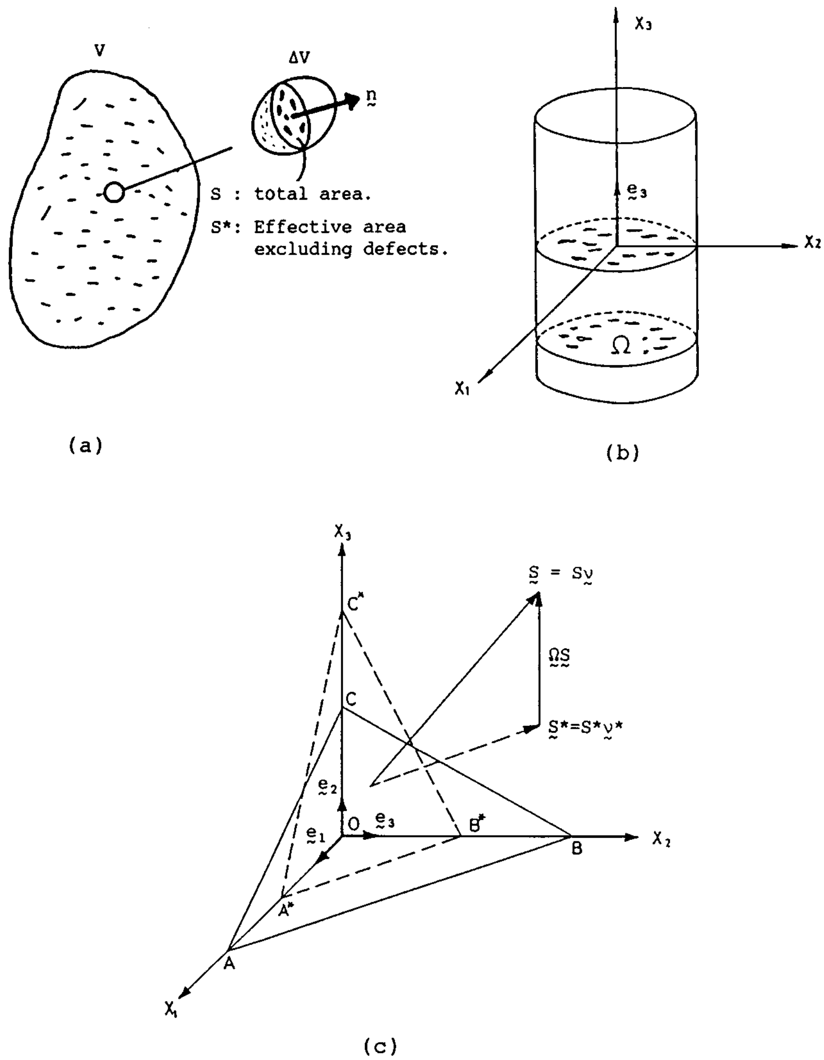

Figure 9. Representation of the damage effect of discontinuities for damage tensor model

shear responses to differentiate the behaviour under tension and compression. The applications of this model to tunnels and underground caverns are described by Kyoya[17] and Kawamoto *et al.*[16]

Swoboda and Ito[42,43] extended this model to model crack propagations in jointed media and gave several examples of its application.

3.4. Micro-structure models

Aydan *et al.*[18,19] proposed two models for discontinuous rock masses based on the micro-structure theory of mechanics.[44] Although the first assumption of Singh is the same as that in this

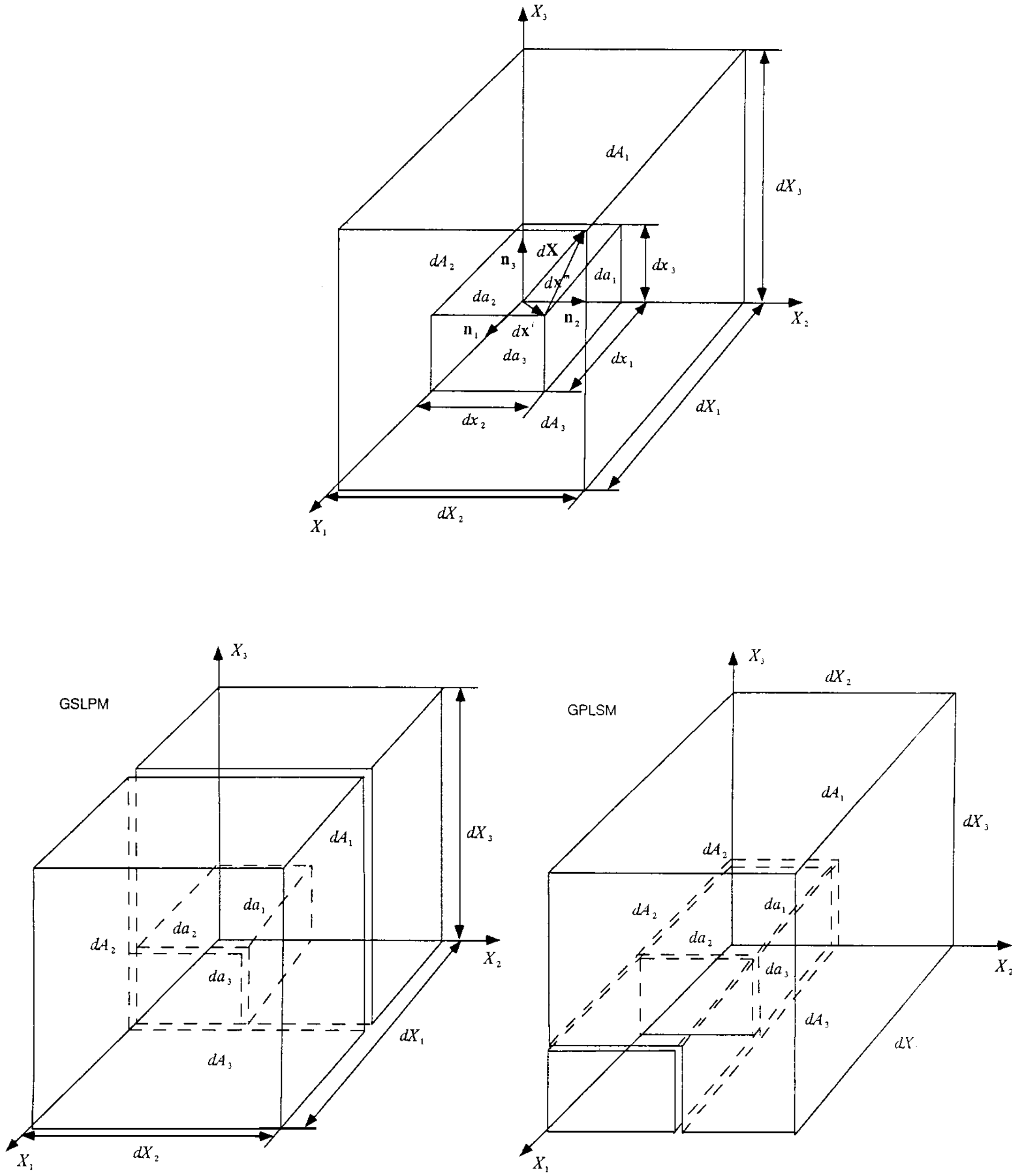

Figure 10. Mechanical representations of the Representative Elementary Volume (REV) for micro-structure models

approach, this model differs from others. The fundamental differences are as follows:

(1) Discontinuities have a finite volume which enables one to model a wide range of discontinuities from joints to faults or fractured zones.
(2) The constitutive law of discontinuities are expressed in the conventional sense of mechanics. In other words, the constitutive law is expressed in terms of stresses and strains and it is uniquely defined.
(3) The constitutive law is not restricted to elasticity and it can be of any kind that can describe the mechanical response of discontinuities.
(4) Stress and strain fields of each constitutient are related to each other using two concepts, namely, Globally Series and Locally Parallel Model: (GSLPM) and Globally Parallel and Locally Series Model: (GPLSM) (Figure 10).

3.5. Homogenization Technique

The homogenization technique was mainly used to obtain the equivalent characteristics of composites[45,46] has been recently applied to soil[47] and rocks[20] (Figure 11). Assumptions 1–3 of the micro-structure model also hold for this technique. Stress and strain fields of constitutients are obtained from a perturbation of the displacement field. An influence tensor, which is a gradient of six vectorial functions called *characteristic deformation functions* for a given representative elementary volume (unit cell) is used to establish relations between the homogenised elasticity tensor and those of its constitutients. Except for very simple cases, the equivalent parameters are obtained using a numerical method such as FEM.

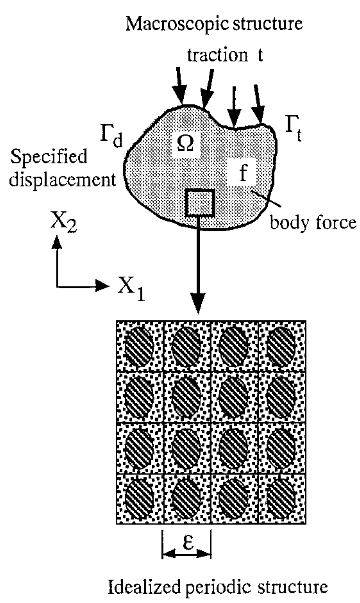

Figure 11. Mechanical representation of a periodic structure for homogenization model

4. HYBRID MODELS

The general tendency in recent years is to use the techniques described above hybridly. In other words, it is a coupling of one of the equivalent models with one of the discrete models. The far field, in which no permanent movement of discontinuities is expected, is modelled by one of the equivalent models. The near field, in which permanent movements of discontinuities are expected, is modelled by one of discrete techniques. Dowding *et al.*[48] described a hybrid model in which the far field and the near field were modelled by the finite element method and the discrete element method, respectively. Lorig *et al.*[49] proposed hybrid scheme of the discrete element method and the boundary element method. Carter and Xiao[50] described a coupled finite element and boundary element scheme in which a jointed rock mass is modelled by the equivalent model of Amadei and Goodman.[40]

5. CONCLUDING REMARKS

A brief but comprehensive review of numerical analysis of tunnels in discontinuous media is given. From this review, the following remarks may be made:

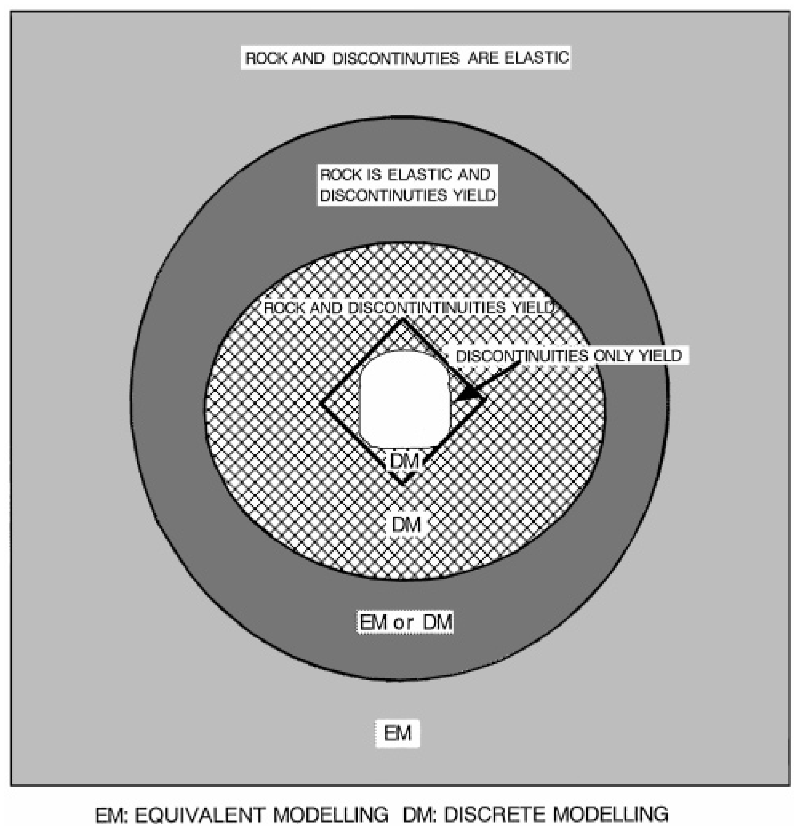

Figure 12. Conceptional illustration of how to select a numerical analysis model for modelling rock masses around a tunnel

(1) Every equivalent model has its own merits and demerits. The selection will generally depend upon the information available on the mechanical and geometrical characteristics of discontinuities existing in the rock masses. The use of equivalent models for non-elastic behaviour of discontinuities may be very cumbersome to use and difficult to validate in practice.
(2) Distinct Element Method (DEM) or Discontinuous Deformation Analysis method (DDA) may be recommended to be used when one is concerned with what happens after failure. However, it should be noted that computational time may not correspond to real time since damping imposed on integrating the equation system in the time domain is not related to actual rate-dependent characteristics of solid blocks and discontinuities.
(3) In practice, it would be desirable to use equivalent continuum models together with discontinuum models hybridly unless one is concerned with the post-failure state of the tunnel (Figure 12). In such a case, some guidelines suggesting how to designate the domain for discontinuum models are necessary. Such guidelines may be obtained from an elastic equivalent analysis of tunnels in which yielding of discontinuities against the sliding and separation around tunnels is assessed. Some methods such as the Discrete Finite Element Method (DFEM) may be promising in this respect as it handles discontinuum and continuum simultaneously within the same solution scheme.

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